Probability of Ray Position in Beam Waveguides

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Abstract—**For the case of infinitely large apertures the following probability distributions are derived in this paper: the probability of finding a ray at the *n*th lens at a distance *r* from the axis, the probability of finding a ray with an amplitude *A* at lens *n*, the cumulative probability of finding a ray with a displacement less than *r* and the cumulative probability of finding a ray with an amplitude less than *A*.**

The case of lenses with finite apertures was explored with the help of computer simulated experiments whose results are given. These experiments show that the probability distributions for the ray amplitudes which were derived for infinite apertures are still useful even in the case of lenses with finite apertures as long as the probability of losing the ray is less than 20 percent.

Introduction

THIS PAPER deals with the statistics of light rays propagating in beam waveguides. A beam waveguide, first proposed by Goubau [1], consists of a sequence of lenses which are capable of guiding a light beam. A beam of electromagnetic radiation tends to spread apart by diffraction. The lenses of the beam waveguide refocus the light beam periodically thus counteracting diffraction and keeping it collimated.

The description of the beam waveguide used in this paper is based on ray optics [2], [3]. A ray optics description of the light beam in a beam waveguide is justified as long as the lens apertures are much larger than the transverse dimension of the light beam. If the light beam illuminates an appreciable portion of each lens, our ray optics treatment is no longer applicable. Ray optics shows that a light beam travels through a beam waveguide following an undulating trajectory. If the lenses of the waveguide are perfectly aligned the ray trajectory is given by the equation

$$r_n = A \cos (n\theta + \phi).$$

The symbols in this equation have the following meaning: r_n is the distance of the ray from the optical axis of the lens system measured at the nth lens, θ is a parameter which depends on the construction of the beam waveguide and will be defined later. A and ϕ are the amplitude and phase of the ray. The ray amplitude A has nothing to do with the light intensity of the ray but is the maximum deviation from the optical axis which the ray can assume as it follows its oscillatory trajectory through the beam waveguide. It is apparent from the equation that r_n may not always assume the value A during one cycle of the ray undulation. Only if the ray trajectory is phased such that $n\theta+\phi=m\pi$ (with m being an integer) does $|r_n|=A$. The values of A and ϕ depend on the initial condition of the ray trajectory. If the lenses of the beam waveguide are not perfectly aligned the ray will follow a more complicated trajectory.

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Several papers [2]–[6] have been published describing the dependence of the rms displacement $\sigma_n=\sqrt{\langle r_n^2\rangle}$ of light rays in a beam waveguide [1] as functions of the statistics of lens displacements. (The symbol $\langle\ \rangle$ indicates an ensemble average.) The knowledge of the rms displacement of the light beam from its trajectory in a perfectly aligned beam waveguide is not quite sufficient to predict the performance of light transmission through such a device. In addition to the rms displacement, one needs to know the probability distribution of the light rays in order to predict where a light ray is likely to penetrate the lens, or what the chances are to have a light ray miss the lens completely, and thus get lost.

Several probability distributions are derived in this paper for the case of lenses with infinitely large apertures. In case the lens apertures can no longer be considered infinite, one can still formulate an integral-difference equation for the joint probability of ray position and angle. This equation is useless, however, since it cannot be solved economically even with the help of an electronic computer. To gain information for the case of lenses with finite apertures, I conducted a series of computer simulated experiments. The result of these experiments indicates that the cumulative probability distribution for the ray amplitudes, derived for the case of infinitely large apertures, remains useful even if the apertures are finite. This holds as long as the total probability, that a ray reaches the lens at which we want to observe it, is larger than 80 percent.

The only parameter entering the probability distributions is the rms deviation σ_n of the light beam at the nth lens. The results of this paper show that the lens aperture $2a$ has to be six times as large as σ_n to ensure that the light ray reaches the nth lens with 99 percent probability.

Probability Distribution for the Beam Position

We consider a beam waveguide composed of ideal, thin lenses. Each lens with lens number ν is displaced by an amount S_ν from a straight line, Fig. 1. Limiting ourselves to the two-dimensional case, we can describe the deviation r_n of the light beam from the straight line by the two simultaneous equations (Fig. 1).

$$r_{n+1} = r_n + L\alpha_n \tag{1a}$$

$$\alpha_{n+1} = \alpha_n - \frac{r_{n+1} - S_{n+1}}{f}. \tag{1b}$$

Equations (1a) and (1b) are the paraxial ray equations. They hold in the limit of small angles α_n as long as the approximation $\tan \alpha_n \approx \alpha_n$ is valid. Elimination of α_n from these equations leads to [2]

$$r_{n+2} - (2 - \kappa)r_{n+1} + r_n = \kappa S_{n+1} \tag{2}$$

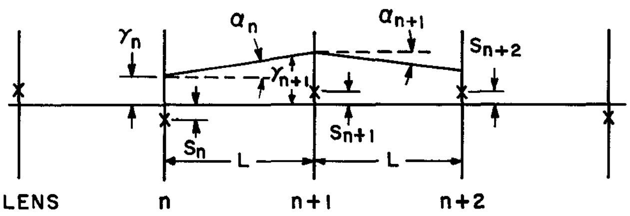

Fig. 1. Beam waveguide with displaced lenses S_n = lens displacement, r_n = beam displacement, α_n = beam angle.

with

$$\kappa = \frac{L}{f},$$

L = lens spacing
f = focal length.

For the light ray which traverses the first two lenses on axis $r_0 = r_1 = 0$, (1) is solved by [2]

$$r_n = \frac{\kappa}{\sin\theta} \sum_{\nu=1}^{n-1} S_\nu \sin(n-\nu)\theta \tag{3}$$

with

$$\cos\theta = 1 - \tfrac{1}{2}\kappa. \tag{4}$$

If each lens displacement S_ν is statistically independent of the others with vanishing average value, $\langle S_\nu \rangle = 0$, and if n is very large, then, according to the central limit theorem [4] the probability distribution to find the light beam at r_n in the interval dr is

$$p_n(r_n)dr = \frac{1}{\sqrt{2\pi}\,\sigma_n} e^{-r_n^2/2\sigma_n^2} dr \tag{5}$$

with

$$\sigma_n^2 = \sum_{\nu=1}^{n-1} \sigma_\nu^2. \tag{6}$$

The standard deviation σ_ν of each term in (3) is given by

$$\sigma_\nu = \delta \frac{\kappa}{\sin\theta} \sin(n-\nu)\theta \tag{7}$$

with δ being the standard deviation of the random variables S_ν which was assumed independent of the index ν. Substituting (7) into (6) and carrying out the summation one obtains

$$\sigma_n = \delta \frac{\kappa}{\sin\theta} \cdot \frac{1}{\sqrt{2}} \sqrt{n - \frac{1}{2} - \frac{\sin(2n-1)\theta}{2\sin\theta}} \cdot \tag{8}$$

Equation (8) is exactly the same as (16) in Hirano and Fukatsu [2], if we express $\sin\theta$ in terms of κ with the help of (4).

It can easily be shown that (5) is true also for small numbers of n, if the lens displacements S_ν are distributed according to a Gaussian probability distribution. Equation (8) for σ_n holds true in this case for all values of n.

We can argue that the probability distribution $p_n(r_n)$ will still be Gaussian, even if the displacements of neighboring lenses are correlated, if these correlations are of finite length. This assumption holds, for example, if the beam waveguide contains many random bends which are uncorrelated among each other. This assumption was made previously by this author [3] and rms values Δ for the beam amplitudes were calculated in that paper. If the bends are truly random, and if many of them are present, then (3) consists of a sum of many random variables each now being a sum of several of the individual terms occurring in (3). The central limit theorem thus holds, and (5) is the proper probability distribution for the beam position. The quantity σ_n can be obtained from a previous paper [3] for several interesting cases using the relation

$$\sigma_n = \frac{1}{\sqrt{2}} \Delta. \tag{9}$$

The factor $\sqrt{2}$ has to be introduced since Δ was the rms value of the beam amplitude while σ_n is the rms value of the actual beam position.

Probability Distribution for the Beam Amplitudes

It has been mentioned already in the introduction that the light ray in a beam waveguide follows an oscillatory trajectory. If the lenses are perfectly aligned (all $S_\nu = 0$), (2) has the solution

$$r_n = A\cos(n\theta + \phi). \tag{10}$$

A is the ray amplitude and ϕ its phase. If the lenses are slightly displaced the trajectory is still given by (10) except that the ray amplitude A is no longer constant, but slowly varying compared to $\cos n\theta$. Writing (3) in the form

$$r_n = \frac{\kappa}{\sin\theta} \left\{ \left(\sum_{\nu=1}^{n-1} S_\nu \cos\nu\theta \right) \sin n\theta - \left(\sum_{\nu=1}^{n-1} S_\nu \sin\nu\theta \right) \cos n\theta \right\} \tag{11}$$

we see that the slowly varying amplitude is given by

$$A = \frac{\kappa}{\sin\theta} \sqrt{ \left(\sum_{\nu=1}^{n-1} S_\nu \cos\nu\theta \right)^2 + \left(\sum_{\nu=1}^{n-1} S_\nu \sin\nu\theta \right)^2 }. \tag{12}$$

Rather than describing the statistics of light rays in terms of the ray position r_n, we might consider a description in terms of the ray amplitude A.

The function $p_n(r_n)$ describes the probability of finding a ray at r_n. This ray may have a small amplitude so that it just reaches up to r_n, or it may have a large amplitude and may reach r_n only because it was ready to cross the axis of the beam waveguide. If the beam waveguide is limited by lens apertures, rays with large amplitudes will be interrupted and unable to continue traveling in the beam waveguide. The probability distribution $p_n(r_n)$ can thus be expected to be altered considerably by the presence of the irises.

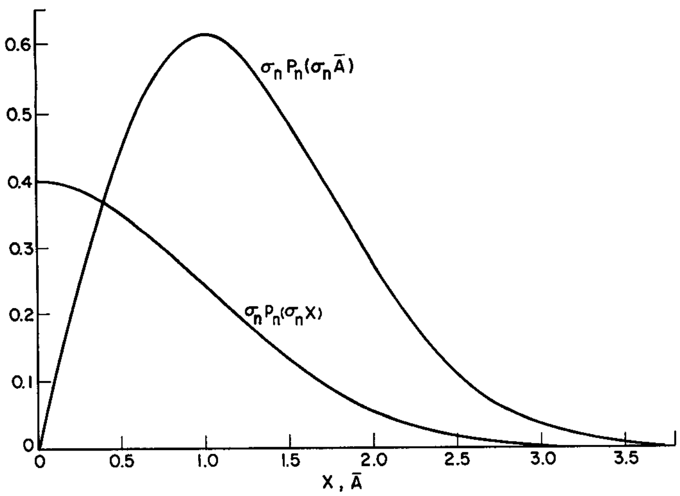

Fig. 2. Probability distributions for beam position $p_n(\sigma_n x)$ and beam amplitude $P_n(\sigma_n A)$ as function of $x = r_n/\sigma_n$ and $\bar{A} = A/\sigma_n$, respectively. σ_n = rms beam deviation.

This consideration suggests that it may be advantageous to use a probability distribution for the ray amplitudes instead of one for the ray position, since one would expect to find it altered only slightly even if irises cut off the rays with the largest amplitudes. An approximate probability distribution $P_n(A)$ for the ray amplitude A at the nth lens can be obtained from $p_n(r_n)$ with the help of an integral equation in the following way. If a ray has the amplitude A the probability to find it at $r_n \leq A$ can be computed from (10) if we assume that the phase ϕ is equally distributed between zero and π with the probability density

$$\Phi(\phi) = \frac{1}{\pi} \cdot \tag{13}$$

The conditional probability distribution $G_A(r_n)$ to find the ray at r_n, if it is known to have the amplitude A, is given by

$$G_A(r_n) = -\left(\frac{dr_n}{d\phi}\right)^{-1} \cdot \Phi(\phi) \tag{14}$$

or with the help of (10)

$$G_A(r_n) = \begin{cases} \dfrac{1}{\pi\sqrt{A^2 - r_n^2}} & r_n < A \\ 0 & r_n > A. \end{cases} \tag{15}$$

The probability $p_n(r_n)$ is thus obtained from the equation

$$p_n(r_n) = \int_0^\infty P_n(A) G_A(r_n) dA \tag{16}$$

or

$$p_n(r_n) = \frac{1}{\pi} \int_{r_n}^\infty P_n(A) \frac{dA}{\sqrt{A^2 - r_n^2}} \cdot \tag{17}$$

The left-hand side of this equation is given by (5) while $P_n(A)$ under the integral sign is not known. It can be shown by substitution into (17) that the solution is a Rayleigh distribution,

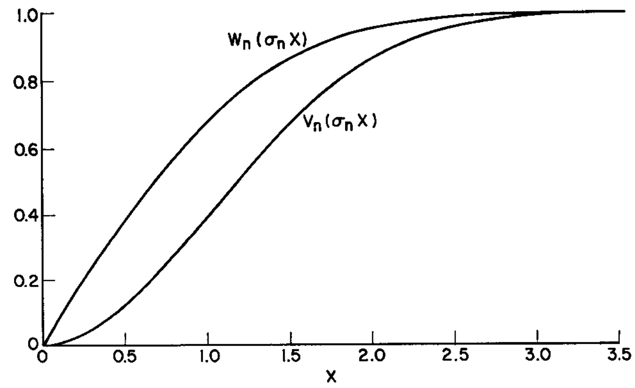

Fig. 3. Cumulative probability for beam position W_n and beam amplitude V_n.

$$P_n(A_n) = \frac{A}{\sigma_n^2} e^{-A^2/2\sigma_n^2}. \tag{18}$$

The result just obtained is well known in noise theory. Its derivation was given since the integral equation (17) will be needed later on. The functions $\sigma_n p_n(\sigma_n x)$ and $\sigma_n P_n(\sigma_n x)$ are shown in Fig. 2. The probability to find a ray with amplitude $A = 0$ is zero while the probability $p_n(r_n)$ reaches its maximum at $r_n = 0$. Looking at Fig. 2, it is well to keep in mind that $p_n(r_n)$ is defined for $-\infty \leq r \leq +\infty$ while $P_n(A_n)$ is defined for $0 \leq A_n \leq \infty$ only.

Cumulative Probability Distributions

For most applications it is more important to know the probability of finding the ray at value less than $|r_n|$. This probability is given by

$$W_n(r_n) = 2\int_0^{r_n} p_n(x) dx. \tag{19}$$

Using the usual definition of the error function, this cumulative probability is given by

$$W_n(r_n) = \text{erf}\left(\frac{r_n}{\sqrt{2}\,\sigma_n}\right). \tag{20}$$

The corresponding cumulative distribution for the ray amplitude A is

$$V_n(A) = \int_0^A P_n(x) dx \tag{21}$$

or, using (18)

$$V_n(A) = 1 - e^{-A^2/2\sigma_n^2}. \tag{22}$$

The functions $W_n(\sigma_n x)$ and $V_n(\sigma_n x)$ are shown in Fig. 3. It is apparent that the cumulative probability for A is always less than that for r_n because only rays with amplitudes not exceeding the value A contribute to V_n, while rays of all amplitudes contribute to $W_n(r_n)$.

Beam Waveguide with Finite Apertures

So far, all our discussions have been concerned with beam waveguides with infinite lens apertures. The case of finite apertures is very much harder to analyze. We might hope that probability distributions such as $V_n(A)$ will still give useful results if the apertures do not cut too deeply into the distribution.

It is easy to formulate an integral-difference equation for the joint distribution $Q_n(r_n, \alpha_n)$, where α_n is the ray angle of (1)

$$Q_{n+1}(r_{n+1}, \alpha_{n+1}) = \int_{-\infty}^{\infty} Q_n(r_n, \alpha_n) U(S_{n+1}) dS_{n+1}. \quad (23)$$

Here $U(S_{n+1})$ is the probability distribution for the lens displacement S_{n+1}. The ray position r_n and angle α_n have to be expressed by r_{n+1} and α_{n+1} according to

$$r_n = (1 - \kappa) r_{n+1} - L\alpha_{n+1} + \kappa S_{n+1} \quad (24a)$$

$$L\alpha_n = \kappa r_{n+1} + L\alpha_{n+1} - \kappa S_{n+1}. \quad (24b)$$

Equation (24) is obtained by solving (1) for r_n and α_n.

Equations (24) and (23) solve the problem in principle. The initial distribution $Q_0(r_0, \alpha_0)$ can be assumed as the product of two δ functions. However, even a computer solution of (23) is too time consuming and therefore out of the question. Equation (23) can be solved analytically for infinite lens apertures, but if the lenses have finite apertures their presence has to be taken into consideration making the analytic solution of (23) a very complicated task.

It is much less time consuming to make a simulated experiment on the computer. I simulated 1000 different, apertured beam waveguides with the same statistics and traced a ray through each guide with the help of (1) starting on the axis at lens 0 and 1. By dropping the rays which hit the lens apertures anywhere along the line and counting the number of rays which arrive with an amplitude less than r_n the cumulative probability $W_n(r_n)$ can be established. The result of this computer experiment is shown in Fig. 4. The solid curves in this figure were obtained by tracing a ray through each of 1000 waveguides, each composed of 100 randomly displaced lenses.

The x-axis is the ratio of r_n/σ_n with σ_n of (8). W_n is the ratio of the number of rays reaching the end of the waveguides at a distance less than r_n, divided by 1000, the total number of experiments. The curves end at the value of x which corresponds to a/σ_n with $2a$ being the width of the lens apertures. The experimental curve labeled $a/\sigma_n = \infty$ is in excellent agreement with the theoretical curve of Fig. 3. It is apparent that the probability to have a ray reach the end of the waveguide with finite lens apertures is considerably reduced from the probability for infinite aperture. The probability for apertured lenses cannot be obtained from the cumulative probability of ray positions for lenses with infinite apertures. For example, one might assume to reach the end of the waveguide at a position $x < 1.77$ with 93 percent probability judging from the curve for lenses with infinite apertures. Instead, we find from the curve labeled $a/\sigma_n = 1.77$ that the probability is only 76 percent.

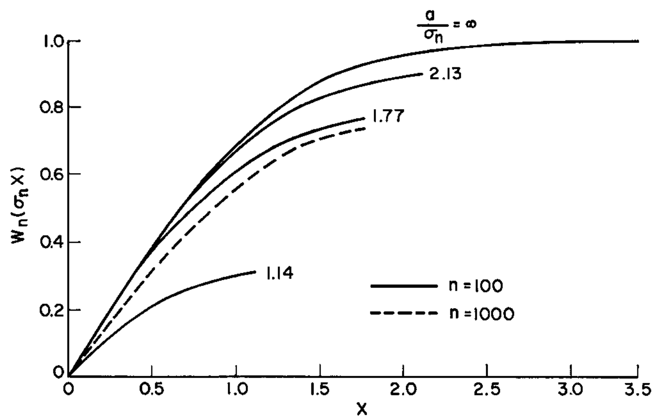

Fig. 4. Cumulative probability for beam position W_n for various values of the ratio a/σ_n, a=aperture half width, σ_n=rms beam deviation.

The dotted curve in Fig. 4 also computed for $a/\sigma_n = 1.77$ is the result of 1000 experiments with waveguides composed of 1000 lenses. The curve is a little lower than the corresponding curve obtained from waveguides 100 lenses long. The difference is not very great, however, so that the normalized representation using $x = r_n/\sigma_n$, which transfers the dependence on the number n of lenses to the parameter σ_n, has a certain approximate validity.

The determination of the ray amplitudes from the computer experiment is complicated by the fact that the value A is not necessarily obtained by any of the values of r_n, as was already mentioned in the Introduction. However, the curves of Fig. 4 can be used to compute the cumulative probability V_n for the ray amplitudes. The integral equation (17) relates the probability p_n of the ray position to P_n the probability of the ray amplitudes. A change of integration variables allows us to write (17) as

$$p_n(r_n) = \frac{1}{\pi} \int_1^{\infty} P_n(ur_n) \frac{du}{u\sqrt{u^2 - 1}} \cdot \quad (25)$$

Substituting (25) into (19), using (21) and changing the integration variable back again, we obtain an integral equation for the cumulative probability of the ray amplitudes

$$W_n(r_n) = \frac{2}{\pi} r_n \int_{r_n}^{\infty} V_n(A) \frac{dA}{A\sqrt{A^2 - r_n^2}} \cdot \quad (26)$$

The cumulative probability has the property

$$V_n(A) = V_n(a) \qquad \text{for } A \geq a$$

with "a" being the lens apertures. This property holds because there are no rays outside the apertures, so that the cumulative probability does not increase beyond $A = a$. This property allows us to rewrite (26)

$$W_n(r_n) + V_n(a) \left[\frac{2}{\pi} \arctan \left(\frac{\sqrt{a^2 - r_n^2}}{r_n} \right) - 1 \right] = \frac{2}{\pi} \int_{r_n}^{a} V_n(A) \frac{dA}{A\sqrt{A^2 - r_n^2}} \cdot \quad (27)$$

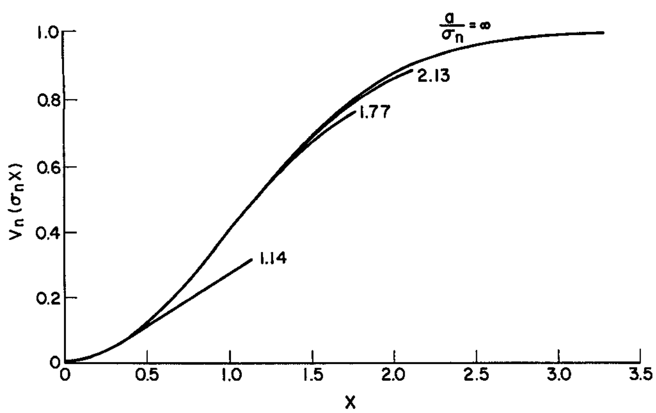

Fig. 5. Cumulative probability for beam amplitude V_n for various values of the ratio a/σ_n, a=aperture half width, σ_n=rms beam deviation.

The total probability for a ray to reach the nth lens is independent of the label used to describe the ray which may be either its position r_n at the nth lens or its amplitude A, so that we may write

$$V_n(a) = W_n(a). \tag{28}$$

Equation (27), amended by (28) can be used to obtain $V_n(A)$ from the curves of Fig. 4 by numerical solution of the integral equation. The result is shown in Fig. 5 where we used again the normalized coordinate $x=A\sigma_n$.

The curves of Fig. 5 confirm the expectation that the cumulative probability V_n is not changed significantly by the presence of apertures as long as the apertures are not too small. The reason for this is that the ray amplitudes tend to increase in the beam waveguide. Decreases in the ray amplitude are sufficiently infrequent so that the presence of apertures does not change the amplitude probability very much.

These results allow us to use (22) to predict the probability that a ray launched on-axis will reach the nth lens, as long as this probability is higher than about 80 percent.

Discussion

The probability distributions allow us to predict the performance of a beam waveguide if σ_n, the rms beam deviation from the waveguide axis, is known. The rms beam deviation is given by (8) for the case of uncorrelated, randomly displaced lenses. For various other more complicated cases σ_n can be obtained from Marcuse [3], Steier [5], and Berreman [6].

The point where either the ray displacement r_n or ray amplitude A equals σ_n is given by $x=1$ in Figs. 3–5. Figure 5, in particular, shows that we need lens apertures with half width $a=3\sigma_n$ if we want to ensure that the light ray reaches the nth lens with 99 percent probability. Assuming $n\gg 1$ and uncorrelated random lens displacements we find with the help of (8) that the ratio of rms lens displacement δ to the half width of the lens apertures "a" has to be

$$\frac{\delta}{a} = \frac{1}{3\sqrt{2}}\sqrt{\frac{4-\kappa}{\kappa}}\,\frac{1}{\sqrt{n}} \tag{29}$$

to ensure a 99 percent probability that the light ray reaches the nth lens. For a confocal system $\kappa=2$ we get

$$\frac{\delta}{a} = \frac{0.236}{\sqrt{n}} = \begin{cases} 2.36\ 10^{-2} & \text{for } n = 100 \\ 6.47\ 10^{-3} & \text{for } n = 1000 \\ 2.36\ 10^{-3} & \text{for } n = 10\,000. \end{cases}$$

The derivation of the probability of the ray amplitudes for apertured lenses from the curve for lenses with infinite apertures is slight (Fig. 5). For $a/\sigma_n=1.77$, for example, the curve labeled $a/\sigma_n=1.77$ gives $V_n=0.765$ while the curve labled $a/\sigma_n=\infty$ gives $V_n=0.81$. Or, expressed differently, to ensure a probability of 76.5 percent beam transmission we would find a required aperture $a/\sigma_n=1.77$ while judging from the curve labeled $a/\sigma_n=\infty$, or equivalently from (22), we would have selected $a/\sigma_n=0.166$ as the necessary ratio.

The cumulative probability $V_n(A)$ of (22) derived for lenses with infinitely large apertures is a reasonable approximation as long as the predicted probabilities are large.

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